

# The $q$ -deformed Bogoliubov transformations

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We give an approach for  $q$ -deformed Bogoliubov transformations that preserve the  $q$ -commutation relations through the union of a left-right module action together with a  $*$ -operation and promoting the coefficients to operators. Finally, we introduce a Hopf structure when  $q$  is a root of unity.

The Bogoliubov transformations defined in the theory of superconductivity [1, 9], connect different notions of vacuum by defining different sets of annihilation and creation operators, furthermore, they are used as a method to analyze particle creation in quantum field theory [8]. Some comments on the impossibility of a  $q$ -deformed Bogoliubov transformations are found in [5]. In this work we take an alternative approach which makes it possible to develop the  $q$ -deformed Bogoliubov transformations. In this process, we promote the constant Bogoliubov coefficients to operators together with a fusion of a left-right module action with a  $*$ -operation. Inside our knowledge, our approach is the first consistent formulation of the  $q$ -deformed Bogoliubov transformations. Consider the annihilation operator  $A$  and the creation operator  $A^*$  together with the Bogoliubov coefficients operators  $u$  and  $v$ . The  $*$ -operation is an antihomomorphism given by sending  $A$  to  $A^*$  and  $u, v$  to the adjoints and  $q$  to its conjugate  $\bar{q}$ . We denote the  $q$ -bracket by  $[x, y]_q = xy - qyx$ , and the following algebra

$$[u, A]_q = [A, u^*]_q = [v, A]_q = [A, v^*]_q = 0. \quad (1)$$

The Bogoliubov transformation is a new pair of operators

$$T(A) = uA + vA^* \quad \text{and} \quad T(A)^* = A^*u^* + Av^*. \quad (2)$$

The evaluation of the  $q$ -bracket is

$$\begin{aligned} [T(A), T(A)^*]_q &= T(A)T(A)^* - qT(A)^*T(A) \\ &= (uA + vA^*)(A^*u^* + Av^*) - q(A^*u^* + Av^*)(uA + vA^*) \\ &= uAA^*u^* + uA^2v^* + vA^{*2}u^* + vA^*Av^* - q(A^*u^*uA + A^*u^*vA^* + Av^*uA + Av^*vA^*) \\ &= |q|^2uu^*AA^* + q^2uv^*A^2 + \bar{q}^2vu^*A^{*2} + |q|^2vv^*A^*A \\ &\quad - q(u^*uA^*A + u^*vA^{*2} + v^*uA^2 + v^*vAA^*). \end{aligned}$$

This expression can be arranged such that it becomes

$$\begin{aligned} [T(A), T(A)^*]_q &= q(\bar{q}uu^* - v^*v)AA^* + q(\bar{q}vv^* - u^*u)A^*A \\ &\quad + q(quv^* - v^*u)A^2 + (\bar{q}^2vu^* - qu^*v)A^{*2}. \end{aligned} \quad (3)$$

In order to agree with the standard Bogoliubov transformations when  $q = 1$ , we introduce the extended  $q$ -hyperbolic identities defined as

$$u^*u - qvv^* = 1 \quad \text{and} \quad v^*v - quu^* = -\frac{1}{\bar{q}}. \quad (4)$$

Then, eq. (3) becomes

$$[T(A), T(A)^*]_q = [A, A^*]_q + q(quv^* - v^*u)A^2 + (\bar{q}^2vu^* - qu^*v)A^{*2}. \quad (5)$$

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Assuming

$$quv^* - v^*u = 0 \quad \text{and} \quad \bar{q}^2vu^* - qu^*v = 0, \quad (6)$$

which are satisfied only for  $q$  a real number. Thus we get the following theorem.

**Theorem 1.** Suppose  $q$  to be a real number,  $A$  an operator,  $u, v$  coefficients operators and a  $*$ -operation satisfying  $(xy)^* = y^*x^*$  and the  $q$ -commuting conditions for brackets

- $[u, A]_q = [A, u^*]_q = [v, A]_q = [A, v^*]_q = 0$  and  $[v^*, u]_q = 0$ ;
- $u^*u - qvv^* = 1$  and  $v^*v - quu^* = -\frac{1}{q}$ .

Therefore, for the new pair of operators  $T(A) = uA + vA^*$  and  $T(A)^* = A^*u^* + Av^*$ , we get

$$[T(A), T(A)^*]_q = [A, A^*]_q. \quad (7)$$

We can extend the previous result for a set of coefficients. This result can be applied for analyzing the particle creation process of black-holes as has been done in [4, 8].

**Theorem 2.** Suppose  $q$  a real number, commuting operators  $A_k$  ( $A_k A_{-k'} = A_{-k'} A_k$ ), coefficients operators  $u_k, v_{k'}$  ( $v_k v_{k'}^* = v_{k'}^* v_k$ ) with  $k, k'$  in a space of momentum and a  $*$ -operation satisfying  $(xy)^* = y^*x^*$  and the  $q$ -commuting conditions for brackets

- $[A_k, A_{k'}^*]_q = [A_{-k'}, A_{-k}^*]_q$
- $[u_k, A_{k'}]_q = [A_{k'}, u_k^*]_q = [v_k, A_{k'}]_q = [A_{k'}, v_k^*]_q = 0$
- $[v_k^*, u_{k'}]_q = 0$
- $u_k^* u_{k'} - qv_{k'} v_k^* = 1$  and  $v_k^* v_{k'} - qu_{k'} u_k^* = -\frac{1}{q}$ .

Therefore, for the new pair of operators  $T(A_k) = u_k A_k + v_k A_{-k}^*$  and  $T(A_{k'})^* = A_{k'}^* u_{k'}^* + A_{-k'} v_{k'}^*$ , we get

$$[T(A_k), T(A_{k'})^*]_q = [A_k, A_{k'}^*]_q. \quad (8)$$

The consecutive application of Bogoliubov transformations give an associative composition where we are searching the way of preserving the left-right module structure in the sense  $T(xAy) = xT(A)y$ . For example the composition of two and three different Bogoliubov transformations are

$$\begin{aligned} T'T(A) &= (uu' + qvv'^*) A + (uv' + q^*vu'^*) A^*, \\ T''T'T(A) &= (uu'u'' + quv'v''^* + qvu''v'^* + q^2vv''^*u'^*) A + \\ &\quad \left( uu'v'' + q^*uv'u''^* + q^*vv''v'^* + q^{*2}vu''^*u'^* \right) A^*. \end{aligned}$$

The authors do not know an interpretation of this structure in the sense of Manin [7] as in [3]. However, there is a Hopf algebra structure as in [2] which we explain at the end of this paper.

An algebra with product and unit, is a Hopf algebra [6] when it has a coproduct  $\Delta$  and counit  $\varepsilon$  and the algebraic anti-homomorphism antipode  $\gamma$  satisfying the following relations

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta,$$

$$m \circ (\varepsilon \otimes id) \circ \Delta = m \circ (id \otimes \varepsilon) \circ \Delta,$$

$$m \circ (\gamma \otimes id) \circ \Delta = m \circ (id \otimes \gamma) \circ \Delta.$$

We consider the Hopf structure given in [2] for the  $q$ -deformed Heisenberg algebra, but here we take the generators  $A, A^*, u, u^*, v$  and  $v^*$  subject to the relations

- $[u, A]_q = [A, u^*]_q = [v, A]_q = [A, v^*]_q = 0$ ;
- $[A, A^*]_q = AA^* - qA^*A = 1$ ; and
- $u^*u - qvv^* = 1$  and  $v^*v - quu^* = -\frac{1}{q}$ .

For  $q^d = 1$  we assume the identity  $A^d = 1$  (so  $A^{-1} = A^{d-1}$ ) and if the  $*$ -operation is preserved, we should have  $(A^*)^d = 1$ , expression for which the physical interpretation is intriguing. In fact, the previous results would imply some cyclic property (condition) when we apply the operators of creation and annihilation to some defined vacuum. For example, consecutive applications of the creation operator over a vacuum  $|0\rangle$ ,  $d$  times, would leave the vacuum unchanged. Analogous interpretation applies to the annihilation operator. Given the previous conditions, there exists the following Hopf algebra structure

$$\begin{aligned}
\Delta(u) &= u \otimes A + A^{-1} \otimes u, & \Delta(v) &= v \otimes A + A^{-1} \otimes v; \\
\Delta(u^*) &= u^* \otimes A + A^{-1} \otimes u^*, & \Delta(v^*) &= v^* \otimes A + A^{-1} \otimes v^*; \\
\Delta(A) &= A \otimes A, \\
\Delta(A^*) &= A^* \otimes 1 + A^{d-2} \otimes A^* + \frac{1}{1-q} (A^{d-1} \otimes A^{d-1}) [1 \otimes 1 - A^{d-1} \otimes 1 - 1 \otimes A]; \\
\epsilon(u) = \epsilon(u^*) = \epsilon(v) = \epsilon(v^*) &= 0, & \epsilon(A) &= 1, & \epsilon(A^*) &= \frac{1}{1-q}; \\
\gamma(u) &= -q^{-1}u, & \gamma(v) &= -q^{-1}v, & \gamma(u^*) &= -qu^*, & \gamma(v^*) &= -qv^*, \\
\gamma(A) &= A^{d-1}, & \gamma(A^*) &= -A^2A^* + \frac{2A}{1-q}.
\end{aligned}$$

In order to obtain  $\Delta$  and  $\gamma$  for the product of two elements we use that  $\gamma$  is an antialgebra map and the bialgebra property

$$\gamma \circ m = m \circ \tau \circ (\gamma \otimes \gamma),$$

$$\Delta \circ m = (m \otimes m) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta),$$

where  $\tau$  is the twist linear map.

The hyperbolic identities  $u^*u - qvv^* = 1$  and  $v^*v - quu^* = -\frac{1}{q}$  imply the following identity for the double tensor algebra

$$\begin{aligned}
1 \otimes A^2 + A^{-2} \otimes 1 &= q(vA^{-1} \otimes Av^* + A^{-1}v^* \otimes vA) - (u^*A^{-1} \otimes Au + A^{-1}u \otimes u^*A) \\
&= \bar{q}(v^*A^{-1} \otimes Av + A^{-1}v \otimes v^*A) - (uA^{-1} \otimes Au^* + A^{-1}u^* \otimes uA).
\end{aligned}$$

We want to remark that when  $q$  is real we only has a bialgebra structure since the existence of the antipode will imply  $q = \pm 1$ .

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